Area magic squares of order 3

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March 5, 2017

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1 Introduction

A magic square of order 3 consists of 9 numbers $A_{ij} \in \mathbb{N}$ (usually) ,with i,j$\in$ {1,2,3}, such that the sum of the elements in each row, column and diagonal equals the same number S.

A_{11}	A_{12}	A_{13}
A_{21}	A_{22}	A_{23}
A_{31}	A_{32}	A_{33}

1.1: Definition of A_{ij}

c-b	c+(a+b)	c-a
c-(a-b)	c	c+(a-b)
c+a	c-(a+b)	c+b

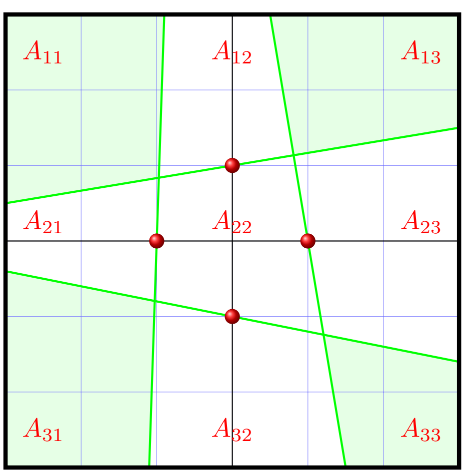
1.2: Pattern, S=3·c

41	113	59
89	71	53
83	29	101

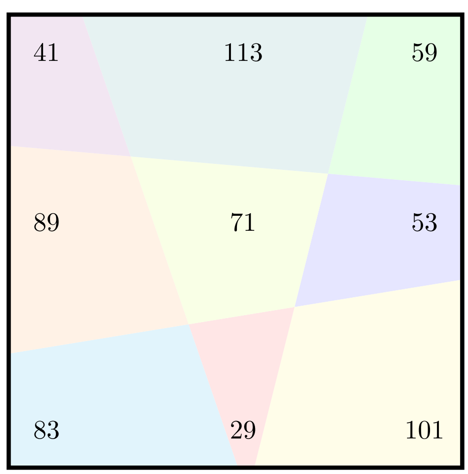
1.3: Example with $A_{ij} \in \mathbb{P}$, S=213

Finding a magic square is not that difficult especially if one uses the pattern given above (1.2). If we search for prime numbers, i.e. $A_{ij} \in \mathbb{P}$, we necessarily have that a,b$\equiv$0 mod 6. In the example (1.3) we have a=12 and b=30, both of which are divisible by 6.

The A_{ij} in figure 1.3 bear no relation to the area of the square in which they are depicted. The aim of this paper is to find a subdivision of the square into 9 regions by using 4 straight dissecting lines, such that each area of a region matches the corresponding value of A_{ij}, given that these numbers constitute a magic square.



1.4: Dissecting lines, fixed points



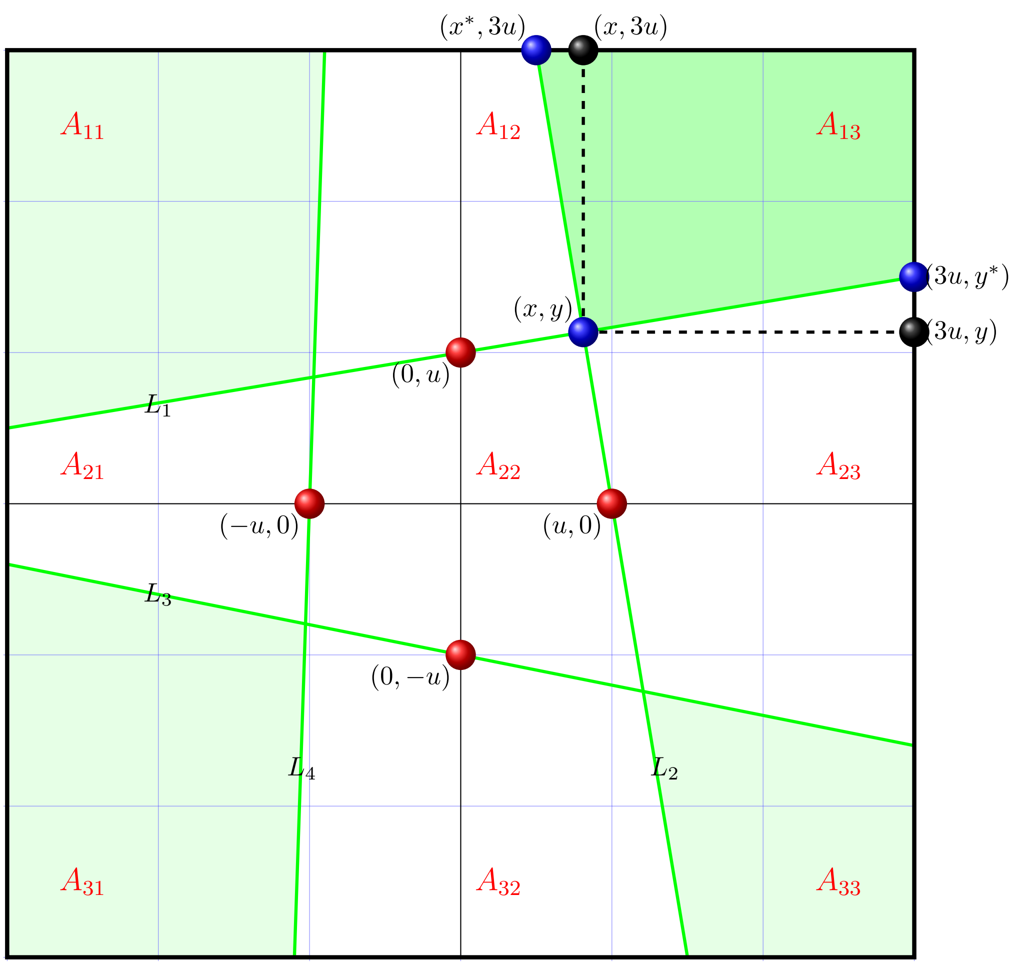
1.5: Area magic prime square

Area magic squares are an idea of William Walkington[1]. In his post on this subject it is shown by Walter Trump that these straight lines have fixed points (shown as red dots in figure 1.4). He found solutions to this problem using the angle of these lines as a parameter. In the same post the results of Francis Gaspalou[2] are found. For each of the lines he used the slope as a parameter and found equations relating A_{11},A_{13},A_{31},A_{33} to the 4 slopes belonging to the dissecting lines (shown as green lines in figure 1.4) and S.

2 Model

Modelling the relation between the areas A_{11},A_{13},A_{31},A_{33} and the parametrisation of the dissecting lines will follow the method employed by Francis Gaspalou, with a few changes:

- The square $[-3u, 3u]\times[-3u, 3u]$ is used, i.e. the origin is in the center of the square. This might facilitate simplification of the expressions involved.
- The inverse of the slope is used as a parameter for the two near vertical lines L_2 and L_4. For the other two lines, L_1 and L_3, the slope is used. This might keep the computation of these parameters numerically stable, especially if $A_{11}\approx A_{31}$ or $A_{13}\approx A_{33}$.



2.1: Model layout

Since the total area of this square is $(6u)^2$ which should equal 3S we have $12u^2$=S.

2.1 Equations of dissecting lines

L_1: $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\u\end{pmatrix}+\alpha\begin{pmatrix}1\\m_1\end{pmatrix}$, L_2: $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}u\\0\end{pmatrix}+\beta\begin{pmatrix}m_2\\1\end{pmatrix}$, L_3: $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\-u\end{pmatrix}+\gamma\begin{pmatrix}1\\m_3\end{pmatrix}$, L_4: $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-u\\0\end{pmatrix}+\delta\begin{pmatrix}m_4\\1\end{pmatrix}$

[1] William Walkington's post on Area Magic Squares and Tori of Order-3
[2] Francis Gaspalou's website on Magic Squares

2.2 Intersection and boundarypoints

Intersection $L_1 \cap L_2$: $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{u}{(1-m_1m_2)} \begin{pmatrix} 1+m_2 \\ 1+m_1 \end{pmatrix}$ and boundary L_1, L_2: $\begin{pmatrix} x* \\ y* \end{pmatrix} = u \begin{pmatrix} 1+3m_2 \\ 1+3m_1 \end{pmatrix}$

Intersection $L_2 \cap L_3$: $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{u}{(1-m_2m_3)} \begin{pmatrix} 1-m_2 \\ -1+m_3 \end{pmatrix}$ and boundary L_2, L_3: $\begin{pmatrix} x* \\ y* \end{pmatrix} = u \begin{pmatrix} 1-3m_2 \\ -1+3m_3 \end{pmatrix}$

Intersection $L_3 \cap L_4$: $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{u}{(1-m_3m_4)} \begin{pmatrix} -1-m_4 \\ -1-m_3 \end{pmatrix}$ and boundary L_3, L_4: $\begin{pmatrix} x* \\ y* \end{pmatrix} = u \begin{pmatrix} -1-3m_4 \\ -1-3m_3 \end{pmatrix}$

Intersection $L_4 \cap L_1$: $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{u}{(1-m_4m_1)} \begin{pmatrix} -1+m_4 \\ 1-m_1 \end{pmatrix}$ and boundary L_4, L_1: $\begin{pmatrix} x* \\ y* \end{pmatrix} = u \begin{pmatrix} -1+3m_4 \\ 1-3m_1 \end{pmatrix}$

2.3 Conditions on the parameters

The lines L_1, L_3 should intersect outside or on the boundary of the main square. The quadrilateral around either A_{21} or A_{23} degenerates into a triangle if the intersection is on the boundary. This will give a condition on m_1, m_3. A similar condition can be derived by comparing the x* values belonging to L_2, L_4.

$|m_1 - m_3| \leq 2/3$ and $|m_2 - m_4| \leq 2/3$

2.4 Calculation of the area

$A_{13} = (3u - x)(3u - y) + \big((x - x*)(3u - y) + (3u - x)(y - y*)\big)/2$

Substitute the values for $x, y, x*, y*$, set u^2=S/12 and multiply by $24(1 - m_1m_2)^2$:

$$24(1-m_1m_2)^2A_{13} = S\big(2(3m_1m_2+m_1-2)(3m_1m_2+m_2-2) - m_1(3m_1m_2+m_2-2)^2 - m_2(3m_1m_2+m_1-2)^2\big)$$

This equation can be written as an expression in m_1m_2 and $m_1 + m_2$:

$$24(1-m_1m_2)^2A_{13} = S\big(-(m_1+m_2)(9(m_1m_2)^2 - 17(m_1m_2) + 8) + 6(m_1m_2)^2 - 14(m_1m_2) + 8\big)$$

This can be further simplified to:

$$24(1-m_1m_2)A_{13} = S\big((m_1+m_2)(9m_1m_2 - 8) + 8 - 6m_1m_2\big) \text{ or}$$
$$24(1-m_1m_2)A_{13} = S\big((m_1+m_2)(1 - 9(1-m_1m_2)) + 2 + 6(1-m_1m_2)\big)$$

This shows that $(1 - m_1m_2)$ is not a factor of the right hand side of the last equation.

2.5 Equations relating area, parameters and magic sum

$$24(1-m_4m_1)^2A_{11} = S\big(2(3m_4m_1-m_4-2)(3m_4m_1-m_1-2) + m_4(3m_4m_1-m_1-2)^2 + m_1(3m_4m_1-m_4-2)^2\big)$$
$$24(1-m_2m_3)^2A_{33} = S\big(2(3m_2m_3-m_2-2)(3m_2m_3-m_3-2) + m_2(3m_2m_3-m_3-2)^2 + m_3(3m_2m_3-m_2-2)^2\big)$$
$$24(1-m_1m_2)^2A_{13} = S\big(2(3m_1m_2+m_1-2)(3m_1m_2+m_2-2) - m_1(3m_1m_2+m_2-2)^2 - m_2(3m_1m_2+m_1-2)^2\big)$$
$$24(1-m_3m_4)^2A_{31} = S\big(2(3m_3m_4+m_3-2)(3m_3m_4+m_4-2) - m_3(3m_3m_4+m_4-2)^2 - m_4(3m_3m_4+m_3-2)^2\big)$$

These equations can be written as expressions in m_im_j and $m_i + m_j$:

$$24(1-m_4m_1)^2A_{11} = S\big(\quad (m_4+m_1)(9(m_4m_1)^2 - 17(m_4m_1) + 8) + 6(m_4m_1)^2 - 14(m_4m_1) + 8\big)$$
$$24(1-m_2m_3)^2A_{33} = S\big(\quad (m_2+m_3)(9(m_2m_3)^2 - 17(m_2m_3) + 8) + 6(m_2m_3)^2 - 14(m_2m_3) + 8\big)$$
$$24(1-m_1m_2)^2A_{13} = S\big(-(m_1+m_2)(9(m_1m_2)^2 - 17(m_1m_2) + 8) + 6(m_1m_2)^2 - 14(m_1m_2) + 8\big)$$
$$24(1-m_3m_4)^2A_{31} = S\big(-(m_3+m_4)(9(m_3m_4)^2 - 17(m_3m_4) + 8) + 6(m_3m_4)^2 - 14(m_3m_4) + 8\big)$$

This can be further simplified to (2.5.1):

$$24(1-m_4m_1)A_{11} = S\big(-(m_4+m_1)(9m_4m_1 - 8) + 8 - 6m_4m_1\big)$$
$$24(1-m_2m_3)A_{33} = S\big(-(m_2+m_3)(9m_2m_3 - 8) + 8 - 6m_2m_3\big)$$
$$24(1-m_1m_2)A_{13} = S\big(\quad (m_1+m_2)(9m_1m_2 - 8) + 8 - 6m_1m_2\big)$$
$$24(1-m_3m_4)A_{31} = S\big(\quad (m_3+m_4)(9m_3m_4 - 8) + 8 - 6m_3m_4\big)$$

2.6 Invariance relations between the parameters

Using $24(A_{11}+A_{33}) = 16S$ and $24(A_{13}+A_{31}) = 16S$ one gets 2 equations relating the m_i. If one adds these we get $24(A_{11}+A_{33}+A_{13}+A_{31}) = 32S$. This relation is also valid if the criterium that the sum of the diagonals is eqaul to S is violated. The main reason for picking this relation is that no area magic square has been found having integer coordinates for its various intersection points, see paragraph 5 .

General invariance relation:

$$\begin{aligned}
&(m_1m_2+m_2m_3+m_3m_4+m_4m_1)\cdot\big(8(m_1m_2m_3m_4)-2(m_1m_2+m_2m_3+m_3m_4+m_4m_1)\big)+\\
&2(m_1m_2+m_2m_3+m_3m_4+m_4m_1)(1-m_1m_2m_3m_4)+\\
&(m_1-m_3)(m_2-m_4)\big(m_1+m_2+m_3+m_4-(m_1m_2m_3+m_2m_3m_4+m_3m_4m_1+m_4m_1m_2)\big)+\\
&2(m_1^2m_2^2+m_2^2m_3^2+m_3^2m_4^2+m_4^2m_1^2) = 8(m_1m_2m_3m_4)^2
\end{aligned} \tag{2.6.1}$$

The good news is that it is a quadratic equation expressed in any of the m_i.

Equation (2.6.1) should be invariant to rotation and reflection. We can check this by either substituting $\{m_2=-m_1, m_3=-m_2, m_4=-m_3, m_1=-m_4\}$ or $\{m_1=-m_1, m_2=-m_4, m_3=-m_3, m_4=-m_2\}$.
By inspection we see that the equation is invariant to these operations.

Invariance relation for one set of parallel lines:

$$m_2=m_4 \implies m_2\big(2m_1m_2m_3-(m_1+m_3)\big)\big(m_2(m_1+m_3-m_1m_2m_3)-1\big)=0$$

The last factor is always negative since $|m_i| \leq \frac{2}{3}$, hence:

$$m_2=m_4 \implies m_2, m_4 = 0 \lor 2m_1m_2m_3 = m_1+m_3$$

If we substitute $m_2, m_4 = 0$ into (2.5.1) and impose $A_{11}+A_{31}$ =S we get $m_1 = m_3$. The second set of dissecting lines is therefore also parallel and we have $A_{11} = A_{33}$, these areas are not distinct, so we arrive at:

$$m_2=m_4 \implies 2m_1m_2m_3 = m_1+m_3 \tag{2.6.2}$$

Two sets of parallel lines are not possible:

$$m_1=m_3, m_2=m_4 \implies m_1(1-m_1m_2)=0$$

As we saw earlier we can't have $m_1, m_3 = 0$, and $m_1m_2 = 1$ is impossible because we have $|m_i| \leq \frac{2}{3}$. Or we could use the argument that necessarily we have $A_{11} = A_{33}$, which are not distinct numbers.

Simplified invariance relation:

Equation (2.6.2) has a much simpler appearance than equation (2.6.1). This equation, (2.6.2), can also be derived from:

$$4m_1m_2m_3m_4 = m_1m_2+m_2m_3+m_3m_4+m_4m_1 \tag{2.6.3}$$

In paragraph 5 solutions are sought when the various intersections points have integer coordinates. Surprisingly most of these solutions follow rule (2.6.3). The first violation is found for S= $12\cdot 336^2$ wich has solution $m_1 = 1/3, m_2 = -5/9, m_3 = -1/6, m_4 = 0$, the second violation is found for S= $12\cdot 504^2$ which has solution $m_1 = 0, m_2 = -1/6, m_3 = -5/9, m_4 = 1/3$.

This suggests trying to rewrite (2.6.1) in a form which either gives equation (2.6.3) or $m_i = 0$ for one value of i. But this is highly speculative, based on two counterexamples. The given examples in paragraph 3, where the m_i are real numbers unequal to 0 and violating (2.6.3) suggest that trying to rewrite the equation is beyond hope.

3 Algorithm

The 4 equations (2.5.3) can be written in the form $F_i(m_1, m_2, m_3, m_4) = 0$, each is a polynomial in two of the m_i and of order 2. It is advised to divide both sides of (2.5.1) by S.

Initialise: $\boldsymbol{m} := \boldsymbol{m_0}$

repeat ($i \geq 0$):
  solve $\boldsymbol{F(m_i)} + \boldsymbol{J(m_i)} \cdot \boldsymbol{dm_i} = \boldsymbol{0}$
  update $\boldsymbol{m_{i+1}} := \boldsymbol{m_i} + \boldsymbol{dm_i}$
until $\|\boldsymbol{dm_i}\|$ is small enough

Boldface characters ($\boldsymbol{F}, \boldsymbol{m}, \boldsymbol{dm}, \boldsymbol{0}$) are chosen to indicate that vectors are implied. $\boldsymbol{J}$ equals the Jacobian of $\boldsymbol{F}$. In a sense this is just an extension to the Newton-Raphson method. We have $\boldsymbol{m_{i+1}} = \boldsymbol{m_i} - \boldsymbol{J(m_i)^{-1}} \cdot \boldsymbol{F(m_i)}$. Instead of computing the inverse of $\boldsymbol{J(m_i)}$ it is recommended to use a $\boldsymbol{LU}$-decomposition. This algorithm might behave badly, because the intital guess $\boldsymbol{m_0}$must be close enough to the final solution in order for this procedure to work, especially if $\boldsymbol{J}$is nearly singular. Note that $\boldsymbol{J(0)}$is singular.

3.1 Initial values

In order to estimate initial values for the m_i we could imagine each area A_{ij} to be square.

$$\widehat{m_1} = \frac{\sqrt{A_{11}} - \sqrt{A_{13}}}{\sqrt{S - A_{11} - A_{13}}}, \widehat{m_2} = \frac{\sqrt{A_{33}} - \sqrt{A_{13}}}{\sqrt{S - A_{33} - A_{13}}}, \widehat{m_3} = \frac{\sqrt{A_{33}} - \sqrt{A_{31}}}{\sqrt{S - A_{33} - A_{31}}}, \widehat{m_4} = \frac{\sqrt{A_{11}} - \sqrt{A_{31}}}{\sqrt{S - A_{11} - A_{31}}} \tag{3.1.1}$$

These can be simplified by writing $3A_{11} = S - \delta$, $3A_{33} = S + \delta$, $3A_{13} = S - \epsilon$ and $3A_{31} = S + \epsilon$.
A zero order approximation in terms of δ and ϵ gives:

$$\widehat{m_1} \approx \frac{3}{2}\frac{A_{11} - A_{13}}{S}, \widehat{m_2} \approx \frac{3}{2}\frac{A_{33} - A_{13}}{S}, \widehat{m_3} \approx \frac{3}{2}\frac{A_{33} - A_{31}}{S}, \widehat{m_4} \approx \frac{3}{2}\frac{A_{11} - A_{31}}{S} \tag{3.1.2}$$

These estimators have the property that $\widehat{m_1} = -\widehat{m_3}$ and $\widehat{m_2} = -\widehat{m_4}$. Substituting into relation (2.6.1) reveals that this approximation is good if $\widehat{m_1}\widehat{m_2}\widehat{m_3}\widehat{m_4}$ is small, i.e. if (in most cases) the A_{ij} are approximately equal and small. These estimators give the approximations:

$$\frac{A_{11}}{A_{33}} \approx \frac{1 + \widehat{m_1} - \widehat{m_2}}{1 + \widehat{m_3} - \widehat{m_4}}, \frac{A_{13}}{A_{31}} \approx \frac{1 - \widehat{m_1} - \widehat{m_2}}{1 - \widehat{m_3} - \widehat{m_4}}$$

3.2 Example

In figure (1.5) the following values are used: $A_{11} = 41, A_{13} = 59, A_{31} = 83, A_{33} = 101$.
We will use (3.1.1) since the A_{ij} are not approximately equal. The Euclidean norm is used.

$\boldsymbol{m_0}$=$(-0.1202, 0.3254, 0.1745, -0.2870)^T$, $\boldsymbol{F_0}$=$(82.3471, -142.3556, 121.2045, -42.9193)^T$,
$\boldsymbol{dm_0}$=$(0.1780, -0.1998, 0.1105, -0.1960)^T$, $\|\boldsymbol{dm_0}\|$ =0.3496
$\boldsymbol{m_1}$=$(0.0578, 0.1255, 0.2849, -0.4830)^T$, $\boldsymbol{F_1}$=$(20.8373, 7.7428, -3.5733, 19.1383)^T$,
$\boldsymbol{dm_1}$=$(-0.0928, 0.0784, -0.0765, 0.0889)^T$, $\|\boldsymbol{dm_1}\|$ =0.1689
$\boldsymbol{m_2}$=$(-0.0351, 0.2040, 0.2084, -0.3941)^T$, $\boldsymbol{F_2}$=$(3.1672, 2.1808, 0.9552, 3.9873)^T$,
$\boldsymbol{dm_2}$=$(-0.0390, 0.0347, -0.0335, 0.0370)^T$, $\|\boldsymbol{dm_2}\|$ =0.0370
$\boldsymbol{m_3}$=$(-0.0741, 0.2387, 0.1749, -0.3571)^T$, $\boldsymbol{F_3}$=$(0.5385, 0.4110, 0.3107, 0.7126)^T$,
$\boldsymbol{dm_3}$=$(-0.0120, 0.0108, -0.0104, 0.0113)^T$, $\|\boldsymbol{dm_3}\|$ =0.0223
$\boldsymbol{m_4}$=$(-0.0861, 0.2495, 0.1646. - 0.3458)^T$, $\boldsymbol{F_4}$=$(0.0494, 0.0389, 0.0326, 0.0683)^T$,
$\boldsymbol{dm_4}$=$(-0.0014, 0.0013, -0.0012, 0.0013)^T$, $\|\boldsymbol{dm_4}\|$ =0.0026
$\boldsymbol{m_5}$=$(-0.0875, 0.2507, 0.1633, -0.3445)^T$, $\boldsymbol{F_5}$=$(0.0007, 0.0005, 0.0005, 0.0010)^T$,
$\boldsymbol{dm_5}$=$(-0.00002, 0.00002, -0.00002, 0.00002)^T$, $\|\boldsymbol{dm_5}\|$ =0.00004

The algorithm behaves poorly, especially at the start. Initially we have $\|\boldsymbol{m_0}\|$ =0.4829 and the adaptation $\|\boldsymbol{dm_0}\|$ =0.3496 which is 72% of the initial norm. Either we need to have much better initial estimates or try to improve on the algorithm itself.

4 Improved algorithm

Before entering the loop we must make sure that the first step doesn't overshoot the final solution. In the hope that the general direction of the adaption $\boldsymbol{dm_0}$ is correct one could try to downsize the stepsize by choosing a small $\boldsymbol{\alpha}$, say 0.10, and compute:

$\boldsymbol{dm_{0^*}} := \dfrac{\boldsymbol{dm_0}}{\|\boldsymbol{dm_0}\|}\boldsymbol{\alpha}\,\|\boldsymbol{m}\|$ as a consequence $\|\boldsymbol{dm_{0^*}}\| = \boldsymbol{\alpha}\,\|\boldsymbol{m}\|$

Initialise: $\boldsymbol{m} := \boldsymbol{m_0}$

solve $\boldsymbol{F(m_0) + J(m_0) \cdot dm_0 = 0}$

norm $\boldsymbol{dm_0} := \dfrac{\boldsymbol{dm_0}}{\|\boldsymbol{dm_0}\|}\boldsymbol{\alpha}\,\|\boldsymbol{m}\|$

update $\boldsymbol{m_1 := m_0 + dm_0}$

repeat ($i > 0$):

solve $\boldsymbol{F(m_i) + J(m_i) \cdot dm_i = 0}$

update $\boldsymbol{m_{i+1} := m_i + dm_i}$

until $\|\boldsymbol{dm_i}\|$ is small enough

4.1 Example revisited

In figure (1.5) the following values are used: $A_{11} = 41, A_{13} = 59, A_{31} = 83, A_{33} = 101$.
We will use (3.1.1) since the A_{ij} are not approximately equal. We'll take $\alpha = 0.1$.

$\boldsymbol{m_0}$=$(-0.1202, 0.3254, 0.1745, -0.2870)^T$, $\boldsymbol{F_0}$=$(82.3471, -142.3556, 121.2045, -42.9193)^T$,
$\boldsymbol{dm_0}$=$(0.1780, -0.1998, 0.1105, -0.1960)^T$, $\|\boldsymbol{dm_0}\|$ =0.3496,$\|\boldsymbol{m_0}\|$ =0.4828,
multiply $\boldsymbol{dm_0}$ by $0.1 \cdot 0.4828/0.3496 = 0.1381$, $\boldsymbol{dm_0}$=$(0.0246, -0.0276, 0.0152, -0.0270)^T$
$\boldsymbol{m_1}$=$(-0.0956, 0.2978, 0.1897, -0.3140)^T$, $\boldsymbol{F_1}$=$(0.3352, -0.5756, 0.4904, -0.1721)^T$,
$\boldsymbol{dm_1}$=$(0.0730, -0.0726, -0.0023, -0.0578)^T$, $\|\boldsymbol{dm_1}\|$ =0.1001
$\boldsymbol{m_2}$=$(-0.0582, 0.2252, 0.1874, -0.3718)^T$, $\boldsymbol{F_2}$=$(0.0116, 0.0095, -0.0058, 0.0065)^T$,
$\boldsymbol{dm_2}$=$(-0.0242, 0.0210, -0.0196, 0.025)^T$, $\|\boldsymbol{dm_2}\|$ =0.0438
$\boldsymbol{m_3}$=$(0.0824, 0.2461, 0.1678, -0.3493)^T$, $\boldsymbol{F_3}$=$(0.0009, 0.0007, 0.0005, 0.0013)^T$,
$\boldsymbol{dm_3}$=$(-0.0049, 0.0044, -0.0042, 0.0046)^T$, $\|\boldsymbol{dm_3}\|$ =0.0091
$\boldsymbol{m_4}$=$(-0.0873, 0.2505, 0.1635, -0.3447)^T$, $\boldsymbol{F_4}$=$(0.00004, 0.00003, 0.00002, 0.00005)^T$,
$\boldsymbol{dm_4}$=$(-0.00024, 0.00022, -0.00021, 0.00023)^T$, $\|\boldsymbol{dm_4}\|$ =0.00045
$\boldsymbol{m_5}$=$(-0.0875, 0.2508, 0.1633, -0.3444)^T$, $\|\boldsymbol{F_5}\|$=$2 \cdot 10^{-7}$, $\|\boldsymbol{dm_5}\|$ =$1 \cdot 10^{-6}$.

4.2 Example palindromic primes

The following values are used[3]: $A_{11} = 10797779701, A_{13} = 12568586521, A_{31} = 12566366521, A_{33} = 14337173341$.
Here $S = 37702429563$. We will use (3.1.2) and $\alpha = 0.1$ and 12 significant digits.

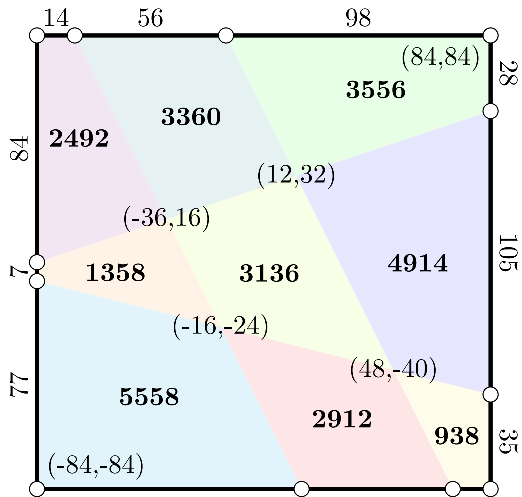
$\boldsymbol{m_5}$=$(-0.0768414564889, 0.0753157136185, 0.0643670825273, -0.0653250521210)^T$,
$\|\boldsymbol{F_5}\|$=$4 \cdot 10^{-20}$, $\|\boldsymbol{dm_5}\|$ =$2 \cdot 10^{-18}$.

[3] Contributed by Carlos Rivera and Jaime Ayala on May 22, 1999 to Harvey Heinz's webpage about Prime Numbers Magic Squares

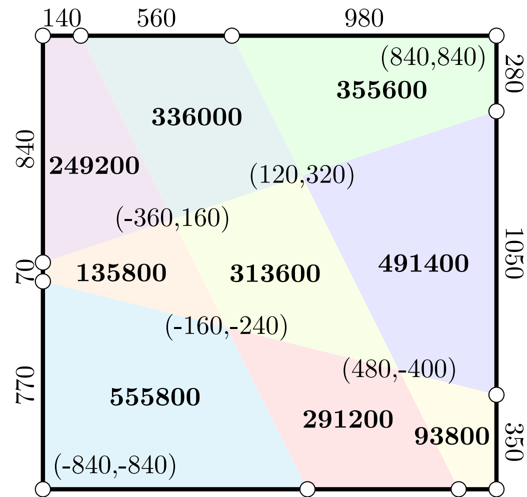
5 Area semi-magic squares with integer coordinates

It would be nice if the coordinates of the polygons are in $\mathbb{Z}$. It is possible to prove that the magic sum must be even. However no solution was found with the property that the diagonal sums where equal to the magic square sum S. All squares do have the property that the sum of both diagonals is equal to 2S, the sum of the elements in the rows and columns do add to S, hence the name area semi-magic squares.

With $S = 12u^2$ all semi-magic squares with $u \leq 300$ have the property that u has at least two factors. No solutions with u odd are found. The solutions are in the set $\{28,56,84,110,112,120,140,168,180,182,196,200,220,224,240,252,280\}$. It helps if u is a highly composite number in order to compute the various intersection points.



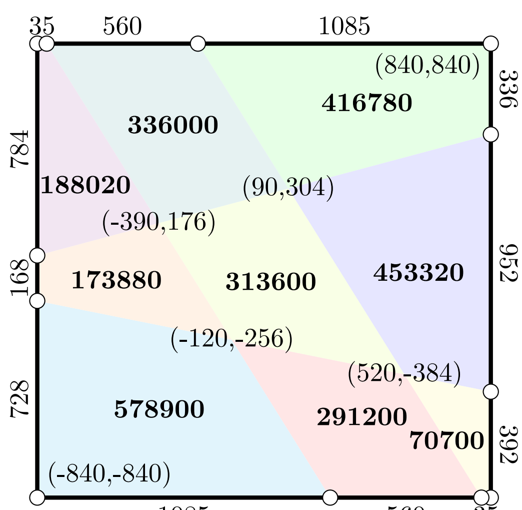
5.1: $S = 12 \cdot 28^2 = 9408$



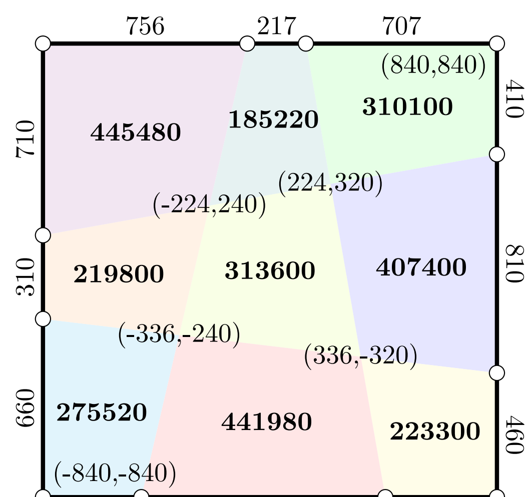
5.2: $S = 12 \cdot 280^2 = 940800$

Most values of S give area semi-magic squares with parallel dissecting lines as shown above. The area semi-magic squares above are congruent, however S=9408 has only 1 solution (modulo symmetries).

If all area semi-magic squares have at least one configuration with parallel lines, one could specify $m_2 = m_4$, choose m_1, m_2 substitute in (2.6.2) and calculate m_3 (speeding up the calculations!).



5.3: $S = 12 \cdot 280^2 = 940800$



5.4: $S = 12 \cdot 280^2 = 940800$

Only $u = 280$ gives a solution where all the distances alongside the border of the square are different (5.4). This solution is the same as previously published by Walter Trump on his site on magic squares[4].

[4] Walter Tump: Linear area nearly magic square of order 3 with integer coordinates

6 Shoelace formula

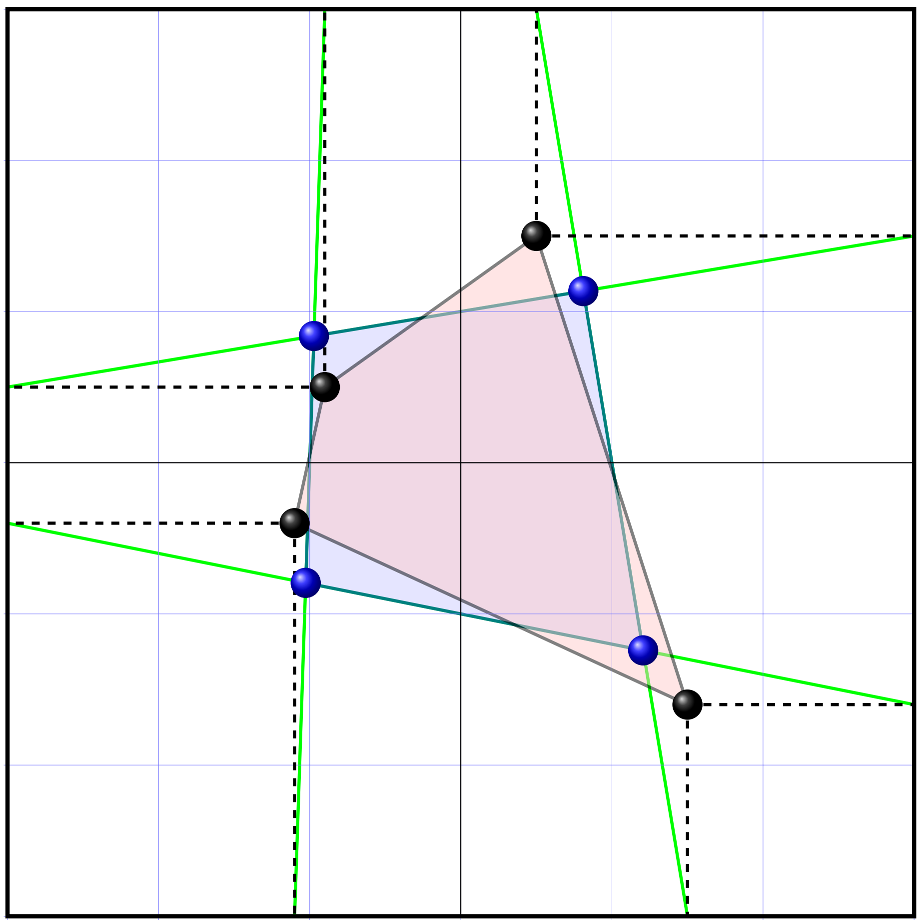
The Shoelace formula[5] relates the area of a non self-intersecting-polygon to the coordinates in a Cartesian plane. Applied to 4 vertices of a polygon (visited counterclockwise, like all mathematicians tend to do 😉):

$$A = \frac{1}{2}\big((x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_1y_4 + x_2y_1 + x_3y_2 + x_4y_3)\big) \quad (6.1)$$

If this formula is applied to the 4 points defined by the coordinates (x^*, y^*) of the boundary points described in section 2.2 we get the following relation assuming $4u^2 = S/3$, a property of area magic squares:

$$A = S\big(\frac{1}{3} - \frac{3}{4}(m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1)\big) \quad (6.2)$$

Which gives a geometric interpretation to part of the general invariance relation (2.6.1), in a sense that it measures the deviation in area from the central quadrilateral.



6.1: Geometric interpretation of $m_1m_2 + m_2m_3 + m_3m_4 + m_4m_1$

The blue area equals $S/3$ (assuming an area magic square), the red area equals A. Upon reordering:

$$m_1m_2+m_2m_3+m_3m_4+m_4m_1 = \frac{4}{9}\Big(1-\frac{A}{S/3}\Big) \quad (6.3)$$

If we drop the link with area magic squares (but retain the fixed points):

$$m_1m_2+m_2m_3+m_3m_4+m_4m_1 = \frac{4}{9}\Big(1-\frac{A}{T/9}\Big) \text{ with } T \text{ the total area of the square} \quad (6.4)$$

If $m_1 = -m_3$ or $m_2 = -m_4$ we have $A = T/9$.

[5] Wikipedia's article on the Shoelace formula