Conjecture 95

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A proof of Conjecture 95. Pascal triangle (2,1), two Conjectures

1 The $(2,1)$ Pascal triangle

The $(\mathbf{2},\mathbf{1})$ Pascal triangle is a visual representation of the coefficients of the polynomial

$$F_n(x) = (\mathbf{2}+\mathbf{1}x)(1+x)^{n-1} = (1+x)^n + (1+x)^{n-1}$$

A polynomial of degree n. One may expand the right hand side and find

$$F_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=0}^{n-1} x^k = \sum_{k=0}^{n-1} \left(\binom{n}{k} + \binom{n-1}{k} \right) x^k + x^n$$

As such we find the coefficients of x^k, with $k < n$ as

$$F_{n,k} = [x^k]F_n(x) = \binom{n}{k} + \binom{n-1}{k}$$

and $F_{n,n} = 1$. In particular we always have $F_{n,0} = 2$ and $F_{n,n-1} = n+1$.

A visualisation, where the first row is indexed by $n = 0$

				k			
n	0	1	2	3	4	5	6
0	1						
1	2	1					
2	2	3	1				
3	2	5	4	1			
4	2	7	9	5	1		
5	2	9	16	14	6	1	
6	2	11	25	30	20	7	1

From the table one may observe that we have $F_{n+1,k} = F_{n,k} + F_{n,k-1}$ if $1 \le k \le n$, similar to the $(1,1)$ Pascal Triangle.

2 A change of perspective

In order to facilitate the problem statement, I'll first introduce

$$G_n(x) = F_{n-1}(x) - x^{n-1}$$

A polynomial of degree $n-2$, where the index n is shifted. I also disposed of the coefficients that are previously displayed on the main diagonal, which are all equal to 1. We are thus left with

$$G_{n,k} = [x^k]G_n(x) = \binom{n-1}{k} + \binom{n-2}{k}$$

where we may pick $n \ge 2$, $0 \le k \le n-2$. A small table to show the effect

	k					
n	0	1	2	3	4	5
2	2					
3	2	3				
4	2	5	4			
5	2	7	9	5		
6	2	9	16	14	6	
7	2	11	25	30	20	7

I would like to state the problem relative to the coefficients with the largest index, it is beneficial to set $m+k=n-2$, or $k=n-2-m$ and define

$$H_{n,m}=G_{n,k}=\binom{n-1}{n-2-m}+\binom{n-2}{n-2-m} \tag{I}$$

In particular we have $H_{n,0}=n-1+1=n$. These numbers are the coefficients of the polynomial

$H_n(x)=x^{n-2}G_n(1/x)$

And we find

	m					
n	0	1	2	3	4	5
2	2					
3	3	2				
4	4	5	2			
5	5	9	7	2		
6	6	14	16	9	2	
7	7	20	30	25	11	2

A slight adaptation would eliminate the coefficients on the main diagonal, but that's not needed in what is to follow.

3 The conjectures

Conjecture 1: If n is prime then $\forall m,\ 0\le m\le n-2:\ H_{n,m}+(-1)^{m+1}m\equiv 0 \mod H_{n,0}$

A table where $H_{n,m}$ is replaced with $H_{n,m}+(-1)^{m+1}m$ for the rows with prime index n

	m					
n	0	1	2	3	4	5
2	2					
3	3	3				
5	5	10	5	5		
7	7	21	28	28	7	7

All the coefficients in row n are divisible by $H_{n,0}=n$.

Conjecture 2: If n is an odd prime then $\forall m,\ 0\le m\le \frac{n-3}{2}:\ H_{n-m,m}\equiv 0 \mod H_{n,0}$

A table which illustrates $n=7$

	m					
n	0	1	2	3	4	5
5	5	9	**7**	2		
6	6	**14**	16	9	2	
7	**7**	20	30	25	11	2

All the bold coefficients are divisible by $H_{n,0}=n=7$.

4 Proof of conjecture 2

Theorem

$$\binom{a}{b} + \binom{a-1}{b} = \frac{(a-1)!}{b!(a-b)!}(2a-b) \tag{4.1}$$

Proof

$$\binom{a}{b} + \binom{a-1}{b} = \frac{a!}{b!(a-b)!} + \frac{(a-1)!}{b!(a-b-1)!} = \frac{(a-1)!}{b!(a-b)!}(a+(a-b))$$

Use the definition of $H_{n,m}$, (I), apply (4.1)

$$H_{n-m,m} = \binom{n-m-1}{n-2m-2} + \binom{n-m-2}{n-2m-2} = n\frac{(n-m-2)!}{(n-2(m+1))!(m+1)!}$$

Since $0 \le m \le (n-3)/2$ with n prime, we have $1 \le m+1 \le (n-1)/2$. Similarly we find $2 \le 2(m+1) \le n-1$ and thus $1 \le n-(2m+1) \le n-2$. If n is prime all values a for which $1 \le a \le n-1$ are relative prime to n. In other words no factor of the two terms in the denominator $(n-2(m+1))!(m+1)!)$ divides n. We find that $H_{n,0} = n$ is a factor of $H_{n-m,m}$.

5 Proof of conjecture 1

Theorem, for prime p we have

$$k!(p-1-k)! \mod p = \begin{cases} -1 & k \text{ even} \\ 1 & k \text{ odd} \end{cases} \tag{5.1}$$

Proof

$$k!(p-1-k)! = \frac{1}{p-1}\frac{2}{p-2}\cdots\frac{k}{p-k}\cdot(p-1)!$$

By Wilson's theorem we have $(p-1)! \equiv -1 \mod p$ and since $p-k \equiv -k \mod p$ we have that every fraction is equivalent to $-1 \mod p$ as well. The result follows.

Use the definition of $H_{n,m}$, (I), where $n = p$, with p prime and reuse (4.1)

$$H_{p,m} = \binom{p-1}{p-2-m} + \binom{p-2}{p-2-m} = \frac{(p-2)!}{(p-2-m)!(m+1)!}(p+m)$$

By Wilson's theorem we have $(p-2)! \equiv 1 \mod p$. For the denominator, where we have $p-2-m+m+1 = p-1$, we may use theorem (5.1) if we set $m+1 = k$. We must correct for a different parity and find

$$H_{p,m} \equiv \frac{(p-2)!}{(p-2-m)!(m+1)!}m \mod p = \begin{cases} m & m \text{ even} \\ -m & m \text{ odd} \end{cases}$$

And the result follows.

6 Closing remark

It would be interesting to find a proof by using the generating function $H_n(x)$.